MULTIPLICITY OF FILTERED RINGS

TOMARI, MASATAKA

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MULTIPLICITY OF FILTERED RINGS*

筑波大学　数学系　　泊　昌孝　(MASATAKA TOMARI)

§1 Introduction and results.

(1.1) Let (V, p) be a germ of a projective variety at a closed point p. It is a fundamental problem to study the ring theoretic properties of the local ring $O_{V,p}$ by means of resolution of singularities ; $\psi : (\tilde{V}, E) \to (V, p)$. In the case $\dim V = 2$, Artin's fundamental cycle for the resolution ψ is important and gives many information of singularities. Let Z_0 be Artin's fundamental cycle for ψ. For example the degree $(Z_0)^2$ is independent of choice of the resolution ψ and we have the relation

$$-(Z_0)^2 \leq \text{the multiplicity of}(V, p)$$

(Ph. Wagreich cf. [Wagreich]). Unfortunately no higher dimensional analogue of this object are studied . In this note we will study the multiplicity of singularities by filtered blowing-ups. We prove an inequality (1.6) which gives a lower bound of multiplicity by the data of tangent cone of the filtration.

An application of our results to a purely elliptic singularity of special type ([IW][Y][T1]) will be given in another note for the talk of "COMMUTATIVE RING THEORY ; JAPAN NO.11".

(1.2) Throughout this note we will fix the following situation. Our singularity (V, p) or local ring $(A, m) = (O_{(V,p)}, m)$ is always assumed as the material coming from some scheme over a field k. Further we will assume (A, m) is analytically unramified after (1.6). In particular,

(A, m) : $d-$dimensional Noether local ring over a field k,

$F = \{F^k\}_{k \geq 0}$: a filtration of ideals as follows ;

($F^0 = A \supset F^1 = m$, $F^k \supset F^{k+1}$, $F^k . F^j \subset F^{k+j}$,

(*) This is a preliminary version. 内容の変化にともない,講演時 "Multiplicity of normal graded rings" より title を上記のものに変えました。

$R = \oplus_{l \geq 0} F^l.T^l \subset A[T]$ is a finitely generated $A-$algebra, where T is an indeterminate.

There is an integer $N > 0$ with $(F^N)^m = F^{Nm}$ for $m \geq 0$.

F^N : $m-$primary)

By these assumption, $G_+ = \oplus_{l \geq 1} F^l/F^{l+1}$ is the homogeneous maximal ideal of G.

Problem (1.2.1). Study the multiplicity $e(m, A)$ of (A, m) from the associated graded ring $gr_F A = G = \oplus_{k \geq 0} F^k/F^{k+1}$ and compare the integers $e(m, A)$ and $e(G_+, G)$.

(We hope that these are very near when G is a "*good*" ring.)

First we shall prove the following .

FACT (1.3). *Let the situation be as above. Then*

$$l(A/m^{l+1}) \leq l(G/(G_+)^{l+1}) \quad \text{for} \quad l \geq 0.$$

In particular we obtain the relations $e(m, A) \leq e(G_+, G)$ *and* *embdim* $A \leq$ *embdim* G .

Proof. The induced filtration on A/m^{l+1} by $F = \{F^k\}$ is given as follows:

$$\begin{array}{ccccccccc} 0 & \to & m^{l+1} & \longrightarrow & A & \longrightarrow & A/m^{l+1} & \to & 0 \\ & & \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \\ 0 & \to & m^{l+1} \cap F^k & \to & F^k & \to & F^k(A/m^{l+1}) & \to & 0 \end{array}$$

Hence we obtain $gr_F(A/m^{l+1}) = gr_F(A)/gr_F(m^{l+1})$. Here we see

$$\begin{aligned} gr_F(m^{l+1}) &= \oplus_{k \geq 0} m^{l+1} \cap F^k / m^{l+1} \cap F^{k+1} \\ &\cong \oplus_{k \geq 0} \frac{F^k \cap m^{l+1} + F^{k+1}}{F^{k+1}}, \\ (G_+)^{l+1} &= \oplus_{k \geq l+1} \frac{\sum_{m_1 + \ldots + m_{l+1} = k, m_i \geq 1} F^{m_1} \ldots F^{m_{l+1}} + F^{k+1}}{F^{k+1}}. \end{aligned}$$

Clearly we have

$$\sum_{m_1 + \ldots + m_{l+1} = k, m_i \geq 1} F^{m_1} \ldots F^{m_{l+1}} \subset F^k \cap m^{l+1}.$$

Hence

$$(G_+)^{l+1} \subset gr_F(m^{l+1}).$$

Therefore

$$l(A/m^{l+1}) \leq l(gr_F(A/m^{l+1})) = l(G/gr_F(m^{l+1})) \leq l(G/(G_+)^{l+1}).$$

Example(1.4). We shall introduce a filtration F on the regular local ring $A = k[[x,y,z]]$ with $m = (x,y,z)A$ by means of the associated order function ν as in the following (cf. [Rees]) :

$\nu(x) = \nu(y) = \nu(z) = 1$, and $\nu(x^2+y^2+z^2) = 3(>2)$ with $F^k = \{\alpha \in A \mid \nu(\alpha) \geq k\} \subset A$.

We can easily check that $G = gr_F(A) \cong k[x,y,z,w]/x^2+y^2+z^2$, $e(m,A) = 1$ and $e(G_+, G) = 2$.

In fact we have

$$\begin{aligned}
F^0 &= A \\
F^1 &= m \\
F^2 &= m^2 \\
F^3 &= m^3 + (x^2+y^2+z^2)A \\
F^4 &= m^4 + (x^2+y^2+z^2)m \\
F^5 &= m^5 + (x^2+y^2+z^2)m^2 \\
F^6 &= m^6 + (x^2+y^2+z^2)m^3 + (x^2+y^2+z^2)^2A \\
&\cdots\cdots ,
\end{aligned}$$

hence $x^2+y^2+z^2 \in F^3 - F^4$ can not be represented by $x,y,z \in F^1 - F^2$ in the ring G. To compute G as in the assertion, remark that if we regard A in the form $A \cong k[[x,y,z,w]]/(w-x^2-y^2-z^2)$, then F is the induced filtration from the filtration on $k[[x,y,z,w]]$ by the degree of monomials as $F^k = \{x^a y^b z^c w^d \in k[x,y,z,w] \mid a+b+c+3d \geq k\}A$.

Example(1.5). We introduce the filtration F on the local ring $A\ (= k[[b,c,y,z]]/b^2 + (y^3+z^7+c^{21})c) \cong k[[a,b,c,y,z]]/(a+y^3+z^7+c^{21}, b^2-ac)$ with $m = (a,b,c,y,z)A$ by the order function ν as :

$\nu(a) = \nu(y^3 + z^7 + c^{21}) = 23$, $\nu(b) = 12$, $\nu(c) = 1$,$\nu(y) = 7$ and $\nu(z) = 3$.
Now $G = gr_F(A) \cong k[a,b,c,y,z]/(y^3+z^7+z^{21}, b^2 - ac)$, $e(m,A) = 2$ and $e(G_+, G) = 6$. Further one can see that G is a normal domain .

These examples say that the integers $e(m,A)$ and $e(G_+,G)$ are different ,in general, even if we assume that G is a normal Gorenstein domain. The next is the main result of this note which gives a lower bound of $e(m,A)$ from the data of G.

THEOREM (1.6). *Let the situation be as in (1.2). Further we assume that A is analytically unramified and that k is an infinite field. Let a system of elements $x_1, \dots, x_s \in G_+$ be a minimal homogeneous generator system of G_+ with $\deg x_1 \leq \deg x_2 \leq \dots \leq \deg x_s$ with $s \geq d = \dim A = \dim G$. Then we have the following*
(1)

$$\begin{aligned}\left(\prod_{i=1}^{d} \deg x_i\right) & \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) \\ & \leq_{(i)} e(m,A) \\ & \leq e(G_+,G) \leq_{(ii)} (\deg x_s)^d \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda).\end{aligned}$$

where $P(G,\lambda) = \sum_{k\geq 0} l(G_k)\lambda^k \in \mathbb{Z}[[\lambda]]$.
(2) If the equality holds in (i), then $e(m,A) = e(G_+,G)$ and there is a parameter system $y_1, \dots, y_d$ of A whose initial form gives a homogeneous parameter system $in(y_1), \dots, in(y_d)$ of G such that $\deg in(y_i) = \deg x_i$ for $i = 1, \dots, d$.
(3) If the equality holds in (ii) and G is normal with $G.C.D.(\deg X_1, \dots, \deg x_s) = 1$, then $e(m,A) = e(G_+,G)$ and G is a homogeneous ring. That is $\deg x_i = 1$ holds for $i = 1, \dots, s$.

In general we have the following.

Remark (1.7) (1) Let $R = R(E,D)$ be a normal d-dimensional graded ring with Demazure's description .

$$D^{d-1} = \lim_{\lambda\to 1}(1-\lambda)^d P(R,\lambda)$$

where $P(R,\lambda) = \sum_{k\geq 0} l(R_k)\lambda^k \in \mathbb{Z}[[\lambda]]$, with $d = \dim R$.

(2) For a graded complete intersection

$$R = k[x_1, \dots, x_{d+s}]/(f_1, \dots, f_s),$$

where $f_1, \ldots, f_s$ is a homogeneous regular sequence of $k[x_1, \ldots, x_{d+s}]$, we have

$$P(R,\lambda) = \frac{(1-\lambda^{degf_1}).\ldots.(1-\lambda^{degf_s})}{(1-\lambda^{degx_1}).\ldots.(1-\lambda^{degx_{d+s}})}.$$

Hence

$$\lim_{\lambda\to 1}(1-\lambda)^d P(R,\lambda) = \frac{(degf_1).\ldots.(degf_s)}{(degx_1).\ldots.(degx_{d+s})}.$$

(1.8) By using (1.7), we will observe (1.4) and (1.5).

(1.8.1) For G of (1.4), we have $degx = degy = degz = 1$ and $degw = 3$. Hence

$$1.1.1.\lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) = 1.1.1.\frac{2}{1.1.1.3} = \frac{2}{3}(\leq 1 = e(m,A)).$$

(1.8.2) For G of (1.5), we have $dega = 23$, $degb = 12$, $degc = 1$, $degy = 7$ and $degz = 3$. Hence

$$1.3.7.\lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) = 1.3.7.\frac{21.24}{1.3.7.12.23} = \frac{42}{23}(\leq 2 = e(m,A)).$$

COROLLARY (1.9). *Let the situation be as in (1.6).*

(1) If the condition

$$\text{the round up of the number } \left(\prod_{i=1}^{d} degx_i\right)\lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) = e(G_+,G)$$

holds, then the equality $e(m,A) = e(G_+,G)$ holds.

(2) If G is a hypersurface with the isolated singularity at G_+, then $e(m,A) = e(G_+,G)$.

Proof . (1) is obvious from (1) of Theorem (1.6). (2) Let us represent G as $G = k[x_1, \ldots, x_{d+1}]/f$ with $degf = h$ and $degX_i = q_i$. Let us represent f by a linear combination of monomials of the form $x^M = \prod_{i=1}^{s} x_i^{m_i}$ with $m_i \geq 0$ as

$$f = \sum_{M\in\left(\mathbb{Z}_{\geq 0}\right)^{d+1}} a_M x^M \quad \text{with } a_M \in k$$

We define the Newton support of f by

$$Support(f) = \{M \in \left(\mathbb{Z}_{\geq 0}\right)^{d+1} \mid a_M \neq 0\}$$

The condition $\sum_{i=1}^{s} q_i m_i = h$ implies $\frac{h}{q_{d+1}} \leq \sum_{i=1}^{s} m_i \leq \frac{h}{q_1}$. Hence we have $q_1 \cdot \ldots \cdot q_d \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda) = \frac{h}{q_{d+1}} \leq$ multiplicity of f = minimum of $\sum_{i=1}^{d+1} m_i$ for $x^M \in Support(f) \leq \frac{h}{q_1} = q_2 \cdot \ldots \cdot q_{d+1} D^{d-1}$. Since $\{f=0\}$ has only isolated singularity at o , a monomial of form $x_i^{m_i} x_{j(i)}$ with $j(i) \in \{1, \ldots, d+1\}$ is contained in $Support(f)$ for each i (K. Saito [S1], V.I. Arnold, P. Orlik- Ph. Wagreich, and Fletcher[Fletcher]). In particular $x_{d+1}^{[h/q_{d+1}]} x_{j(d+1)} \in Support(f)$. Hence the multiplicity of f equals the round up of the rational number $\frac{h}{q_{d+1}}$.

Example (1.10). Let A be ("a normal graded complete intersection ") as follows : $A = k[[x,y,z,w,u]]/(f_1,f_2)$ with the filtration F on A naturally induced as $degx = degy = degz = degw = 1$, $degu = 2$ and $degf_1 = degf_2 = 3$. We have $G = k[x,y,z,w,u]/(f_1,f_2)$ with $degx = degy = degz = degw = 1$, $degu = 2$ and $degf_1 = degf_2 = 3$. By (1.6) we obtain

$$\frac{9}{2} \leq e(m,A) \leq e(G_+,G) \leq 36.$$

Since (A,m) is not a tangential complete intersection with respect to the maximal-ideal-adic filtration on A, the lower bound is the best. But the upper bound of this implication is very bad.

In the rest of this note we give a outline of proof of Theorem (1.6) and state some generalities on the rational number $\left(\prod_{i=1}^{d} degx_i\right) \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda)$ for the normal graded ring R in terms of Demazure's description of R.

§2 The openness of reduction property.

The purpose of this section is to prove (2.7) which we will use in §4.

(2.1) Let (V,p) be a singularity over a field k and $(O_{V,p},m)$ be the associated local ring. We assume that p is a closed point of a projective variety $\bar{V}$ over the field k. Let I be an $m-$primary ideal of $O_{V,p}$. Let $\pi : (\tilde{V},A) \to (V,p)$ be a projective morphism such that $I.O_{\tilde{V}}$ is a locally principal $O_{\tilde{V}}-$module. We will represent $I.O_{\tilde{V}}$ as

$$I.O_{\tilde{V}} = O_{\tilde{V}}(-D(I,\pi))$$

by a Cartier divisor on $\tilde{V}$.

THEOREM (2.2). *Let the situation below as above. Asumme* $d = \dim O_{\tilde{V}}$. *Then* $e(I, O_{V,p}) = (-1)^{d+1}D(I,\pi)^d$.

Proof. Let a projective variety $\bar{V}$ be a compactification of V and $\psi : \bar{V}_1 \to \bar{V}$ be the blowing-up of $\bar{V}$ with center I. There is a natural morphism $\tau : \bar{\tilde{V}} \to \bar{V}_1$ which satisfies the relation $\pi = \psi.\tau$. We have $I^k = \psi_*(I^k O_{\tilde{V}})$ and $R^i\psi_*(I^k O_{\tilde{V}}) = 0$ ($i \geq 1$) for arbitrary large integer k (EGA III). We have $l(I^k/I^{k+1}) = \chi(\bar{V}, I^k/I^{k+1})$. By Leray's spectral sequence

$$E_2^{p,q} = H^p(\bar{V}, R^q\psi_*(I^k O_{\bar{V}_1})) \Longrightarrow_p H^n(\bar{V}_1, I^k O_{\bar{V}_1}),$$

we have

$$\sum_{q\geq 0}(-1)^q\chi(\bar{V}, R^q\psi_*(I^k O_{\bar{V}_1})) = \chi(\bar{V}_1, I^k O_{\bar{V}_1}).$$

Hence for $k >> 0$, we obtain

$$\begin{aligned} l(I^k/I^{k+1}) &= \chi(\bar{V}, I^k O_{\bar{V}}) - \chi(\bar{V}, I^{k+1} O_{\bar{V}}) \\ &= \chi(\bar{V}_1, I^k O_{\bar{V}_1}) - \chi(\bar{V}_1, I^{k+1} O_{\bar{V}_1}). \end{aligned}$$

Let $P \in \mathbb{Q}[t]$ be the Hilbert-Samuel polynomial defined as $P(k) = \chi(\bar{V}_1, I^k O_{\bar{V}_1}/I^{k+1} O_{\bar{V}_1})$ for $k >> 0$[Kl]. We have degree $P = d - 1$. Let us set of polynomials $\Delta^{(m)}P$ for $1 \leq m \leq d$ as ; $\Delta^{(1)}P(k) = P(k) - P(k-1)$, . .. , $\Delta^{(m)}P = \Delta(\Delta^{(m-1)}P)$, inductively. Here $\Delta^{(d-1)}P$ is the constant function $e(I, O_{V,p})$. Further we have

$$\begin{aligned} &\Delta^{(1)}P(k) \\ &= \chi(\bar{V}_1, I^k O_{\bar{V}_1}) - \chi(\bar{V}_1, I^{k+1} O_{\bar{V}_1}) - \chi(\bar{V}_1, I^{k-1} O_{\bar{V}_1}) + \chi(\bar{V}_1, I^k O_{\bar{V}_1}) \\ &= -\chi(\bar{V}_1, I^{k+1} O_{\bar{V}_1}) + 2.\chi(\bar{V}_1, I^k O_{\bar{V}_1}) - \chi(\bar{V}_1, I^{k-1} O_{\bar{V}_1}). \end{aligned}$$

An by similar calculations we obtain

$$\begin{aligned} \Delta^{(d-1)}P(k) &= -\sum_{i=0}^{d}\binom{d}{i}(-1)^i\chi(\bar{V}_1, I^{k+1-i} O_{\bar{V}_1}) \\ &= -\deg_{O_V(-D(I,\psi))}(O_{bar V_1}) \\ &= -(O_{V_1}(-D(I,\psi))^d.O_{V_1})_{V_1} (\text{ the intersection symbol [Kl] }) \\ &= (-1)^{d+1}D(I,\psi)^d. \end{aligned}$$

Since τ is birational, $(-1)^{d+1}D(I,\psi)^d = (-1)^{d+1}D(I,\pi)^d$.
Hence $e(I, O_{V,p}) = (-1)^{d+1}D(I,\pi)^d$. Q.E.D.

We will apply (2.2) to the following.

Let $J \subset I$ be $m-$primary ideals of $O_{V,p}$. Recall J is a reduction of I if there is an integer $r > 0$ such that $I^r J = I^{r+1}$ (Northcott-Rees [NR]).

THEOREM (2.3) (D. REES [HIO][REES], SEE ALSO J. LIPMAN [LIPMAN]). *Assume that (A, m) is analytically unramified. Then J is a reduction of I if and only if the equality $e(J, O_{V,p}) = e(I, O_{V,p})$ holds.*

COROLLARY (2.4). *For $m-$primary ideals $J \subset I$, the following three conditions are equivalent each other.*
(1) The equality $e(J, O_{V,p}) = e(I, O_{V,p})$ holds.
(2) J is a reduction of I.
(3) There exists a birational morphism $\psi : \tilde{V} \to V$ such that the relation $J.O_{\tilde{V}} = I.O_{\tilde{V}}$ holds.

Proof. The equivalence of (1) and (2) are due to (2.3). Assume the condition (3) holds. There exists a birational morphism $\tau : V' \to \tilde{V}$ such that $I.O_{V'} = J.O_{V'}$ is local principal. By (2.2) we have $e(J, O_{V,p}) = e(I, O_{V,p})$. Next we assume there is an integer $r > 0$ such that $I^r J = I^{r+1}$. Let $\varphi : V' \to V$ be a birational morphism such that $I.O_{V'}$ is local principal. Then we have $J.I^r.O_{V'} = I^{r+1}.O_{V'}$ and have $J.O_{V'} = I.O_{V'}$.

By this we can see the reduction property of ideals are open condition as in the following sense.

DEFINITION (2.5). *Let J be an ideal of $O_{V,p}$. A deformation of ideal J $\varrho : \tilde{J} \to Y \ni o$ over a scheme Y with a reference point o is an ideal $\tilde{J}$ of $O_{V \times Y}$ at $p \times Y$ such that $\varrho^{-1}(o) = J$.*

PROPOSITION (2.6). *Let $J \subset I$ be $m-$primary ideals of $O_{V,p}$ and $\varrho : \tilde{J} \to Y \ni o$ be a deformation of ideal of J . Suppose J is a reduction of I. Then there is a Zariski open neighborhood U of o in Y where $\tilde{J}_y = \varrho^{-1}(y)$ is a reduction of I for any point y of U.*

Proof. There is an integer $r > 0$ such that $I^r J = I^{r+1}$ and $\varphi : V' \to V$ be a birational morphism such that $I.O_{V'}$ is local principal and V' is normal. Then we have $J.O_{V'} = I.O_{V'}$. Consider the morphism $\tilde{\varphi} : V' \times Y \to V \times Y$ with $I.O_{V' \times Y} \supset \tilde{J}$. Here $I.O_{V' \times Y}$ is defined as an invertible $O_{V' \times Y}$ in a trivial extension of $I.O_{V'}$. Now we have the relation $\tilde{J}O_{V'\times Y} + \varphi^{-1}(m_o)I.O_{V'\times Y} = I.O_{V'\times Y}$. Hence $I.O_{V'\times Y}$ and $\tilde{J}.O_{V'\times Y}$ are equal at each generic points of $V' \times Y$ which contains a point of $\tilde{\varphi}^{-1}(o) = V' \times o$. In particular the reflexive hull $\left(\tilde{J}.O_{V'\times Y}\right)^{**}$ equals $I.O_{V'\times Y}$. Let $S \subset V' \times Y$ be the non-reflexive locus of $\tilde{J}.O_{V'\times Y}$. Then S does not intersects $V' \times o$ and $\varphi(S)$ does not contain the point o. By Corollary (2.4) at any point $y \in Y - \varphi(S)$, $\tilde{J}_y$ is a reduction of I. Q.E.D.

COROLLARY (2.7). *Let I be an m-primary ideal of $O_{V,p}$ generated as $I = (f_1, \dots, f_s)$. Suppose that $O_{V,p}$ contains a field k and that there is a reduction J of I written as*

$J = (y_1, \dots, y_m)O_{V,p}$ where

$$y_i = \sum_{j=1}^{s} a_{i,j}x_j, \quad \text{with} \quad a_{i,j} \in k \;\; 1 \le i \le m, \;\; 1 \le j \le s.$$

Then there is a Zariski open neighborhood U of $(a_{i,j})$ in k^{sm} such that $J_b = (z_1, \dots, z_m)O_{V,p}$ is a reduction of I for $z_i = \sum_{j=1}^{s} b_{i,j}x_j$, with $(b_{i,j}) \in U$

Proof . Define the deformation of J by $\tilde{J} = \coprod_{b\in k^{sm}} J_b$ over k^{sm}. The the assertion follows from Proposition 3. Q.E.D.

We state the following which is a higher dimensional analogue of a theorem of Laufer (cf. [L1]):

THEOREM (2.8). *Let (W, w) be a normal d-dimensional singularity and $(x_1, \dots, x_d)$ a parameter system of $O_{W,w}$. Let $\psi : X \to W$ be a projective modification with normal X and $E = \psi^{-1}(w)$. We write $div_X(x_iO_X)$ by*

$$div_X(x_iO_X) = D(x_iO_W, \psi) \; + \; W_{x_i,\psi} \quad i = 1, \dots, d,$$

where $W_{x_i,\psi}$ is the strict transform of $\{x_i = 0\}$ and $D(x_iO_W, \psi)$ is the part of E. We assume that the divisor $W_{x_i,\psi}$ is $\mathbb{Q}$-Cartier for i = 1, ... ,d.

If $W_{x_1,\psi} \cap \dots \cap W_{x_d,\psi}$ is empty, we have the relation

$$e((x_1, \dots, x_d), O_{W,w}) = (-1)^{d+1}D(x_1O_W, \psi) \dots D(x_dO_W, \psi)$$

We omit the proof.

§3 On Demazure's description of normal graded rings.

(3.1) The purpose of this section is to collect the generalities of Demazure's description $R(E,D)$ of the normal graded ring R in the connection with the number

$$\left(\prod_{i=1}^{d} deg x_i\right) \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda).$$

As there are many good references on this subjects [D],[W 敬 1],[W 敬 2], we will review a computation method for Demazure's divisor D by a tentative way as follows (cf. [T1]): Let $R = \oplus_{k\geq 0} R_k$ be a normal d-dimensional graded ring with $R_0 = k$, R_+ the homogeneous maximal ideal with a generator consisting in homogeneous elements $x_1, \ldots, x_s$ as $R_+ = (x_1, \ldots, x_s)R$ with $x_i \in R_{q_i}$ for $i = 1, \ldots, s$. We assume the condition G.C.D.$(q_1, \ldots, q_s) = 1$. There are integers $u_1, \ldots, u_s$ such that $\sum_{i=1}^{s} u_i q_i = 1$. We choose a homogeneous element T of the quotient field of R as $T = \Pi_{i=1}^{s}(x_i)^{u_i}$. We represent

$$x_i R = \cap_{Q\in HP(R)} Q^{(a_{iQ})}, \quad i = 1, \ldots, s.$$

Here $HP(R)$ is the set of homogeneous prime ideals of height 1. By Demazure's fundamental works we can represent D as follows:

Theorem (3.2)(Demazure [D]). *In the above situation, we define the divisor D associated to T as*

$$D = \sum_{i=1}^{s}\left(\sum_{Q\in HP(R)} \frac{u_i a_{iQ}}{N(Q)} . V(Q)\right) \in Div(E)\otimes \mathbb{Q}$$

where $V(Q)$ is the integral Weil divisor on $E = Proj(R)$ defined by Q and $N(Q)$ is the the integer defined as ;

$N(Q) = G.C.D.\{n \in \mathbb{Z} \mid n > 0$ and $(R/Q)_n \neq 0\}$ (cf. (5.9.1) of [TW1]).

Then we obtain the equality

$$R = \oplus_{k\geq 0} H^0(E, O_E(kD)).T^k \text{ in } k(E)[T].$$

Example (3.3). Let R is a normal $d-$dimensional ($d \geq 2$) graded ring of the Brieskorn type as follows : $R = \mathbb{C}[x_1, \ldots, x_{d+1}]/\{(x_1)^{a_1} + \ldots + (x_{d+1})^{a_{d+1}}\}$ where

$a_1, ..., a_{d+1}$ are integers ≥ 2. Introduce the weight of each monomials of $\mathbb{C}[x_1, ..., x_{d+1}]$ as the degree of $x_i = L.C.M.(a_1, ..., a_{d+1})/a_i$. We simply denote it as q_i , for $i = 1, \ldots , d+1$.

Then the Demazure divisor D associated to T is written as

$$D = \sum_{i=1}^{d+1} \frac{u_i}{G.C.D.(q_1, ..., \hat{i}, ..., q_{d+1})} .D_i \in Div(E) \otimes \mathbb{Q}$$

where D_i is the integral Weil divisor on $E = \text{Proj}(R)$ defined by the canonical morphism $D_i = \text{Proj}(R/x_iR) \to \text{Proj}(R) = E$ for $i = 1, ..., d+1$ (see (1.9) of [T1] for a proof).

Lemma (3.4). *Let $R = R(E, D)$ be a normal d-dimensional graded ring with Demazure's description . Let us consider the singularity of $Spec(R)$ at $V(R_+)$. Let*

$$\psi : C = C(E, D) = Spec_E(\oplus_{k\geq 0}O_E(kD)) \to Spec(R)$$

be the partial resolution by the filtered blowing-up of $Spec(R)$ with respect to the filtration induced by grading of $Spec(R)$. Let $x_1, \ldots , x_d \in R$ be a parameter system at R_{R_+} . Suppose $x_1, \ldots , x_r$ with $r \leq d$ be homogeneous elements. Then we have

$$\dim W_{x_1,\psi} \cap \ldots \cap W_{x_d,\psi} \leq d - r - 1 \quad in \quad C(E, D).$$

Hence in the case $r = d$, $W_{x_1,\psi} \cap \ldots \cap W_{x_d,\psi}$ is empty . In this case $e((x_1, \ldots , x_d), R_{R_+})$ is computed by Theorem (2.8).

By (2.8) and (3.4) we obtain the following.

Corollary (3.5). *Let $R = R(E, D)$ be a normal d-dimensional graded ring with Demazure's description and $x_1, \ldots , x_d \in R$ be a homogeneous parameter system of R .*

$$e((x_1, \ldots , x_d), R) = (-1)^{d+1} \left(\prod_{i=1}^{d} degx_i\right) .E^d.$$

Here E^d is the intersection multiplicity in $C = C(E, D)$.

Lemma (3.6). *Let $R = R(E, D)$ be a normal d-dimensional graded ring with Demazure's description and*

$$\psi : C = C(E, D) = Spec_E(\oplus_{k\geq 0}O_E(kD)) \to Spec(R)$$

be the filtered blowing-up of $Spec(R)$ with respect to the filtration induced by grading of $Spec(R)$.

Then we have the relation

$$D^{d-1} = (-1)^{d+1} E^d.$$

By using (1.7) we obtain the following.

COROLLARY (3.7). *In the situation (3.6), assume $x_1, \dots, x_d \in R$ be a homogeneous parameter system.*

$$\begin{aligned} e((x_1, \dots, x_d), R) &= \left(\prod_{i=1}^{d} deg x_i\right) .D^{d-1} \\ &= \left(\prod_{i=1}^{d} deg x_i\right) . \lim_{\lambda \to 1} (1-\lambda)^d P(R, \lambda). \end{aligned}$$

§4 Proof of Theorem (1.6).

(4.1) *The inequality (i) of (1)* . Let a system of elements $x_1, \dots, x_s$ of the maximal ideal m of A whose initial forms with respect to the filtration F give the minimal homogeneous generator of G_+ as follows ; $x_i \in F^{q_i} - F^{q_i+1}$ and the initial forms $in_F(x_i)$ $= \bar{x}_i \in G_{q_i}$ satisfies the relations $G_+ = (\bar{x}_1, \dots, \bar{x}_s)G$ and $q_1 \leq \dots, \leq q_s$. We can easily see the relations $m = F^n + (x_1, \dots, x_s)A$ for any positive integer n. There is an integer n such that $F^n \subset m^2$. Hence $m = (x_1, \dots, x_s)$ by NAK.

There is a system of parameter $y_1, \dots, y_d$ which is a minimal reduction of m and given as linear combination of $x_1, \dots, x_s$ as follows :

$$y_i = \sum_{j=1}^{s} a_{i,j} x_j, \quad \text{where} \quad a_{i,j} \in k.$$

with $1 \leq i \leq d, \ 1 \leq j \leq s$. By the openness of reduction property (Corollary (2.7)), we may assume $A = (a_{i,j})_{1 \leq i, j \leq d}$ is regular. So we can choose y_i in the following form:

$$y_i = x_i + \sum_{j=d+1}^{s} a_{i,j} x_j, \quad \text{where} \quad a_{i,j} \in k$$

for $1 \leq i \leq d$ from the beginning .

Let L be a positive integer divided by $L.C.M.(deg x_1, \dots, deg x_d)$. By Leck's lemma

$$e((y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}), A) = \frac{L}{q_1}. \dots .\frac{L}{q_d}.e((y_1, \dots, y_d), A).$$

Since $y_i^{\frac{L}{q_i}} \in F^L$ for $1 \le i \le d$, we have

$$e(F^L, A) \le \frac{L}{q_1}. \dots .\frac{L}{q_d}.e((y_1, \dots, y_d), A).$$

There is an integer L as above and satisfies the relation $F^{mL} = \left(F^L\right)^m$ for any positive integer m, that is $\left(\oplus_{k\ge 0} F^k.T^k\right)^{(L)} = A[F^L.T^L]$. To finish the proof, it is sufficient to show the following:

LEMMA (4.2). *Let L be a positive integer such that the relation $F^{mL} = \left(F^L\right)^m$ holds for any positive integer m. Then*

$$e(F^L, A) = L^d. \lim_{\lambda\to 1}(1-\lambda)^d P(G, \lambda).$$

Proof. By the assumption , $\oplus_{k\ge 0} F^{kL}/F^{(k+1)L}$ is generated by F^L/F^{2L} . Hence we obtain the equality (see §13 and §14 of [Matsumura]):

$$e(F^L, A) = \lim_{\lambda\to 1}(1-\lambda)^d P(\oplus_{k\ge 0} F^{kL}/F^{(k+1)L}, \lambda).$$

Let us introduce the notation as $G^{(L)} = \oplus_{k\ge 0} F^{kL}/F^{kL+1}$ and $G^{(L,l)} = \oplus_{k\ge 0} F^{kL+l}/F^{kL+l+1}$ for $l = 0, \dots, L-1$. Since there is an integer M such that $F^L.F^b = F^{L+b}$ holds for any $b \ge M$, $G^{(L,l)}$ is a finite $G^{(L)}-$module for $l = 0, \dots, L-1$. As graded $G^{(L)}-$modules, we calculate the Poincare series ; $P(G^{(L,l)}, \mu) \in \mathbb{Z}[[\mu]]$ for $l = 0, \dots, L-1$. For each i, $\lim_{\mu\to 1}(1-\mu)^d P(G^{(L,l)}, \mu)$ is a finite number. We have the relations

$$\begin{aligned}(1-\mu)^d P(G,\mu) &= \sum_{l=0}^{L-1}(1-\mu)^d P(G^{(L,l)}, \mu^L).\mu^l \\ &= \sum_{l=0}^{L-1}(1-\mu^L)^d P(G^{(L,l)}, \mu^L).\frac{(1-\mu)^d}{(1-\mu^L)^d} \\ &+ \sum_{l=0}^{L-1}(1-\mu^L)^d P(G^{(L,l)}, \mu^L).\frac{(1-\mu)^d}{(1-\mu^L)^d}(\mu^l - 1).\end{aligned}$$

Hence

$$\begin{aligned}\lim_{\mu\to 1}(1-\mu)^d P(G,\mu) &= \lim_{\mu\to 1}(1-\mu^L)^d \sum_{l=0}^{L-1} P(G^{(L,l)},\mu^L).\frac{1}{L^d}\\ &= \lim_{\mu\to 1}(1-\mu)^d P(\sum_{l=0}^{L-1} G^{(L,l)},\mu).\frac{1}{L^d}\\ &= \lim_{\mu\to 1}(1-\mu)^d P(\bigoplus_{k\geq 0} F^{kL}/F^{(k+1)L},\mu).\frac{1}{L^d}\\ &= e(F^L,A)\frac{1}{L^d}.\end{aligned}$$

(4.3) *Proof* of (2). Let $y_1, \dots, y_d$ be a parameter system of A as in the arguments of (4.1). By the assumption we have the equality

$$e((y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}), A) = e(F^L, A).$$

Hence $(y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}})$ is a reduction of F^L by a Theorem of Rees. There is an integer $r > 0$ such that

$$\left(F^L\right)^{r+1} = \left(F^L\right)^r (y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}) \text{ in } A.$$

Let $\psi : X = Proj(\oplus_{k\geq 0} F^k.T^k) \to Spec(A)$ be the filtered blowing-up of $Spec(A)$ by F. We have

$$\left(F^L\right)^{r+1} O_X = \left(F^L\right)^r (y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}) O_X \text{ in } O_X.$$

Here $R^L O_X = O_X(L)$ is an invertible O_X−module sheaf, we obtain the relation

$$F^L O_X = O_X(L) = (y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}}) O_X.$$

We represent the strict transform of the scheme $Spec(A/y_i)$ by ψ as $W_{y_i,\psi}$ for $i = 1, \dots, d$. Since $(y_1^{\frac{L}{q_1}}, \dots, y_d^{\frac{L}{q_d}})O_X$ is locally free , $W_{y_1,\psi} \cap \dots \cap W_{y_d,\psi} \cap E$ is empty. Here

$$W_{y_j,\psi} \cap E = Proj\,(G/In(y_j)G)\,,$$

where $In(y_j)$ is the initial homogeneous element of y_j. Therefore $In(y_1), \dots, In(y_d)$ is a parameter system of G.

(4.4) *Proof of the inequality (ii) of (1).* There is an integer L satisfies the relation $G|_{mL} = (G|_L)^m$, that is $(G^\natural)^{(L)} = G[G|_L]$. Now we have $e(G|_L, G) = L^d.\lim_{\lambda\to 1}(1-$

$\lambda)^d P(G,\lambda)$ by (4.2). We can easily see $G|_{q_s L} \subset (G_+)^L$. Hence

$$L^d e(G_+,G) = e((G_+)^L,G) \le e(G|_{q_s L},G) = e((G|_L)^{q_s}) = q_s^d e(G|_L,G)$$
$$= q_s^d . L^d . \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda)$$

Therefore $e(G_+,G) \le q_s^d . \lim_{\lambda\to 1}(1-\lambda)^d P(G,\lambda)$.

(4.5)*Proof of (3)* By assumption $(G|_L)^{q_s}$ is a reduction of G_+^L. As same as in the arguments of (1) we consider the filtered blowing up ψ of $Spec(G)$. By the assumption G is described by Demazure's method as $G = R(E,D)$. As in §3, we will represent ψ as :

$$\psi : C = C(E,D) = Spec_E(\oplus_{k\ge 0} O_E(kD)) \to Spec(R).$$

We obtain the relation

$$R_+^L O_C = (R|_L)^{q_s} O_C = O_C(-q_s LE) \text{ on } C.$$

Since x_1^L is not contained in $R|_{Lq_1+1}$, we have the relation $R_+^L O_C = O_C(-Lq_1 E)$. Hence $q_1 = q_s$.

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